To appear

A study of a curious arithmetic function

Bakir FARHI
bakir.farhi@gmail.com

Abstract

In this note, we study the arithmetic function $f : \mathbb{Z}_+^* \to \mathbb{Q}_+^*$ defined by $f(2^k \ell) = \ell^{1-k}$ ($\forall k, \ell \in \mathbb{N}$, ℓ odd). We show several important properties about that function and then we use them to obtain some curious results involving the 2-adic valuation.

MSC: 11A05.

Keywords: Arithmetic functions; Least common multiple; 2-adic valuation.

1 Introduction and notations

The purpose of this paper is to study the arithmetic function $f : \mathbb{Z}_+^* \to \mathbb{Q}_+^*$ defined by:

$$f(2^k \ell) = \ell^{1-k} \qquad (\forall k, \ell \in \mathbb{N}, \ell \text{ odd}).$$

We have for example $f(1) = 1, f(2) = 1, f(3) = 3, f(12) = \frac{1}{3}, f(40) = \frac{1}{25}, \dots$. So it is clear that $f(n)$ is not always an integer. However, we will show in what follows that f satisfies among others the property that the product of the $f(r)$'s ($1 \leq r \leq n$) is always an integer and it is a multiple of all odd prime number not exceeding n. Further, we exploit the properties of f to establish some curious properties concerning the 2-adic valuation.

The study of f requires to introduce the two auxiliary arithmetic functions $g : \mathbb{Q}_+^* \to \mathbb{Z}_+^*$ and $h : \mathbb{Z}_+^* \to \mathbb{Q}_+^*$, defined by:

$$g(x) := \begin{cases} x & \text{if } x \in \mathbb{N} \\ 1 & \text{else} \end{cases} \qquad (\forall x \in \mathbb{Q}_+^*) \tag{1}$$

$$h(r) := \frac{r}{g(\frac{r}{2})g(\frac{r}{4})g(\frac{r}{8})\cdots} \qquad (\forall r \in \mathbb{Z}_+^*) \tag{2}$$

Remark that the product in the denominator of the right-hand side of (2) is actually finite because $g(\frac{r}{2^i}) = 1$ for any sufficiently large i; so h is well-defined.

Some notations and terminologies. Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers. For a given prime number p, we let v_p denote the usual p-adic valuation. We define the *odd part* of a positive rational number α as the positive rational number, denoted $\mathrm{Odd}(\alpha)$, so that we have $\alpha = 2^{v_2(\alpha)} \cdot \mathrm{Odd}(\alpha)$. Finally, we denote by $\lfloor . \rfloor$ the integer-part function and we often use in this paper the following elementary well-known property of that function:

$$\forall a, b \in \mathbb{N}^*, \forall x \in \mathbb{R} : \quad \left\lfloor \frac{\lfloor \frac{x}{a} \rfloor}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor .$$

2 Results and proofs

Theorem 2.1 *Let n be a positive integer. Then the product $\prod_{r=1}^{n} f(r)$ is an integer.*

Proof. For a given $r \in \mathbb{N}^*$, let us write $f(r)$ in terms of $h(r)$. By writing r in the form $r = 2^k \ell$ ($k, \ell \in \mathbb{N}$, ℓ odd), we have by the definition of g:

$$g\left(\frac{r}{2}\right) g\left(\frac{r}{4}\right) g\left(\frac{r}{8}\right) \cdots = \left(2^{k-1}\ell\right)\left(2^{k-2}\ell\right) \times \cdots \times \left(2^0 \ell\right) = 2^{\frac{k(k-1)}{2}} \ell^k .$$

So, it follows that:

$$h(r) := \frac{r}{g(\frac{r}{2}) g(\frac{r}{4}) g(\frac{r}{8}) \cdots} = \frac{2^k \ell}{2^{\frac{k(k-1)}{2}} \ell^k} = 2^{\frac{k(3-k)}{2}} \ell^{1-k} = 2^{\frac{k(3-k)}{2}} f(r).$$

Hence

$$f(r) = 2^{\frac{v_2(r)(v_2(r)-3)}{2}} h(r). \tag{3}$$

Using (3), we get for all $n \in \mathbb{N}^*$:

$$\prod_{r=1}^{n} f(r) = 2^{\sum_{r=1}^{n} \frac{v_2(r)(v_2(r)-3)}{2}} \prod_{r=1}^{n} h(r). \tag{4}$$

By taking the odd part of each of the two hand-sides of this last identity, we obtain:

$$\prod_{r=1}^{n} f(r) = \mathrm{Odd}\left(\prod_{r=1}^{n} h(r)\right) \qquad (\forall n \in \mathbb{N}^*). \tag{5}$$

So, to confirm the statement of the theorem, it suffices to prove that the product $\prod_{r=1}^{n} h(r)$ is an integer for any $n \in \mathbb{N}^*$. To do so, we lean on the following sample property of g:

$$g\left(\frac{1}{a}\right) g\left(\frac{2}{a}\right) \cdots g\left(\frac{r}{a}\right) = \left\lfloor \frac{r}{a} \right\rfloor! \qquad (\forall r, a \in \mathbb{N}^*).$$

Using this, we have:

$$\begin{aligned}
\prod_{r=1}^{n} h(r) &= \prod_{r=1}^{n} \frac{r}{g\left(\frac{r}{2}\right) g\left(\frac{r}{4}\right) g\left(\frac{r}{8}\right) \cdots} \\
&= \frac{n!}{\displaystyle\prod_{r=1}^{n} g\left(\frac{r}{2}\right) \cdot \prod_{r=1}^{n} g\left(\frac{r}{4}\right) \cdot \prod_{r=1}^{n} g\left(\frac{r}{8}\right) \cdots} \\
&= \frac{n!}{\lfloor\frac{n}{2}\rfloor! \lfloor\frac{n}{4}\rfloor! \lfloor\frac{n}{8}\rfloor! \cdots}.
\end{aligned}$$

Hence

$$\prod_{r=1}^{n} h(r) = \frac{n!}{\lfloor\frac{n}{2}\rfloor! \lfloor\frac{n}{4}\rfloor! \lfloor\frac{n}{8}\rfloor! \cdots} \tag{6}$$

(Remark that the product in the denominator of the right-hand side of (6) is actually finite because $\lfloor\frac{n}{2^i}\rfloor = 0$ for any sufficiently large i).
Now, since $\lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{4}\rfloor + \lfloor\frac{n}{8}\rfloor + \cdots \leq \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots = n$ then $\frac{n!}{\lfloor\frac{n}{2}\rfloor! \lfloor\frac{n}{4}\rfloor! \lfloor\frac{n}{8}\rfloor! \cdots}$ is a multiple of the multinomial coefficient $\binom{\lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{4}\rfloor + \lfloor\frac{n}{8}\rfloor + \cdots}{\lfloor\frac{n}{2}\rfloor \ \lfloor\frac{n}{4}\rfloor \ \lfloor\frac{n}{8}\rfloor \ \cdots}$ which is an integer. Consequently $\frac{n!}{\lfloor\frac{n}{2}\rfloor! \lfloor\frac{n}{4}\rfloor! \lfloor\frac{n}{8}\rfloor! \cdots}$ is an integer, which completes this proof. ■

Theorem 2.2 *Let n be a positive integer. Then $\displaystyle\prod_{r=1}^{n} f(r)$ is a multiple of* $\mathrm{Odd}(\mathrm{lcm}(1, 2, \ldots, n))$.
In particular, $\displaystyle\prod_{r=1}^{n} f(r)$ is a multiple of all odd prime number not exceeding n.

Proof. According to the relations (5) and (6) obtained during the proof of Theorem 2.1, it suffices to show that $\frac{n!}{\lfloor\frac{n}{2}\rfloor! \lfloor\frac{n}{4}\rfloor! \lfloor\frac{n}{8}\rfloor! \cdots}$ is a multiple of $\mathrm{lcm}(1, 2, \ldots, n)$. Equivalently, it suffices to prove that for all prime number p, we have:

$$v_p\left(\frac{n!}{\lfloor\frac{n}{2}\rfloor! \lfloor\frac{n}{4}\rfloor! \lfloor\frac{n}{8}\rfloor! \cdots}\right) \geq \alpha_p, \tag{7}$$

where α_p is the p-adic valuation of $\mathrm{lcm}(1, 2, \dots, n)$, that is the greatest power of p not exceeding n. Let us show (7) for a given arbitrary prime number p. Using Legendre's formula (see e.g., [1]), we have:

$$\begin{aligned} v_p\left(\frac{n!}{\lfloor\frac{n}{2}\rfloor!\lfloor\frac{n}{4}\rfloor!\lfloor\frac{n}{8}\rfloor!\cdots}\right) &= \sum_{i=1}^{\infty}\left\lfloor\frac{n}{p^i}\right\rfloor - \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left\lfloor\frac{n}{2^j p^i}\right\rfloor \\ &= \sum_{i=1}^{\alpha_p}\left(\left\lfloor\frac{n}{p^i}\right\rfloor - \sum_{j=1}^{\alpha_2}\left\lfloor\frac{n}{2^j p^i}\right\rfloor\right) \qquad (8) \end{aligned}$$

Next, for all $i \in \{1, 2, \dots, \alpha_p\}$, we have:

$$\sum_{j=1}^{\alpha_2}\left\lfloor\frac{n}{2^j p^i}\right\rfloor = \sum_{j=1}^{\alpha_2}\left\lfloor\frac{\lfloor\frac{n}{p^i}\rfloor}{2^j}\right\rfloor \leq \sum_{j=1}^{\alpha_2}\frac{\lfloor\frac{n}{p^i}\rfloor}{2^j} < \left\lfloor\frac{n}{p^i}\right\rfloor.$$

But since $\left(\lfloor\frac{n}{p^i}\rfloor - \sum_{j=1}^{\alpha_2}\lfloor\frac{n}{2^j p^i}\rfloor\right)$ $(i \in \{1, 2, \dots, \alpha_p\})$ is an integer, it follows that:

$$\left\lfloor\frac{n}{p^i}\right\rfloor - \sum_{j=1}^{\alpha_2}\left\lfloor\frac{n}{2^j p^i}\right\rfloor \geq 1 \qquad (\forall i \in \{1, 2, \dots, \alpha_p\}).$$

By inserting those last inequalities in (8), we finally obtain:

$$v_p\left(\frac{n!}{\lfloor\frac{n}{2}\rfloor!\lfloor\frac{n}{4}\rfloor!\lfloor\frac{n}{8}\rfloor!\cdots}\right) \geq \alpha_p,$$

which confirms (7) and completes this proof. ■

Theorem 2.3 *For all positive integer n, we have:*

$$\prod_{r=1}^{n} h(r) \leq \mathrm{c}^n,$$

where $c = 4.01055487\dots$.
In addition, the inequality becomes an equality for $n = 1023 = 2^{10} - 1$.

Proof. First, we use the relation (6) to prove by induction on n that:

$$\prod_{r=1}^{n} h(r) \leq n^{\log_2 n} 4^n \qquad (9)$$

- For $n = 1$, (9) is clearly true.
- For a given $n \geq 2$, suppose that (9) is true for all positive integer $< n$

and let us show that (9) is also true for n. To do so, we distinguish the two following cases:
1st case: (if n is even, that is $n = 2m$ for some $m \in \mathbb{N}^*$).
In this case, by using (6) and the induction hypothesis, we have:

$$\begin{aligned}\prod_{r=1}^{n} h(r) &= \binom{2m}{m}\prod_{r=1}^{m} h(r)\\ &\leq \binom{2m}{m} m^{\log_2 m} 4^m\\ &\leq m^{\log_2 m} 4^{2m} \qquad \left(\text{since } \binom{2m}{m} \leq 4^m\right)\\ &\leq n^{\log_2 n} 4^n,\end{aligned}$$

as claimed.
2nd case: (if n is odd, that is $n = 2m+1$ for some $m \in \mathbb{N}^*$).
By using (6) and the induction hypothesis, we have:

$$\begin{aligned}\prod_{r=1}^{n} h(r) &= (2m+1)\binom{2m}{m}\prod_{r=1}^{m} h(r)\\ &\leq (2m+1)\binom{2m}{m} m^{\log_2 m} 4^m\\ &\leq m^{\log_2 m+1} 4^{2m+1} \qquad \left(\text{since } 2m+1 \leq 4m \text{ and } \binom{2m}{m} \leq 4^m\right)\\ &\leq n^{\log_2 n} 4^n,\end{aligned}$$

as claimed.
The inequality (9) thus holds for all positive integer n. Now, to establish the inequality of the theorem, we proceed as follows:
— For $n \leq 70000$, we simply verify the truth of the inequality in question (by using the Visual Basic language for example).
— For $n > 70000$, it is easy to see that $n^{\log_2 n} \leq (\mathrm{c}/4)^n$ and by inserting this in (9), the inequality of the theorem follows.
The proof is complete. ■

Now, since any positive integer n satisfies $\prod_{r=1}^{n} f(r) \leq \prod_{r=1}^{n} h(r)$ (according to (5) and the fact that $\prod_{r=1}^{n} h(r)$ is an integer), then we immediately derive from Theorem 2.3 the following:

Corollary 2.4 *For all positive integer n, we have:*

$$\prod_{r=1}^{n} f(r) \leq \mathrm{c}^n,$$

where c *is the constant given in Theorem 2.3.* ■

To improve Corollary 2.4, we propose the following optimal conjecture which is very probably true but it seems difficult to prove or disprove it!

Conjecture 2.5 *For all positive integer n, we have:*

$$\prod_{r=1}^{n} f(r) \ < \ 4^n.$$

Using the Visual Basic language, we have checked the validity of Conjecture 2.5 up to $n = 100000$. Further, by using elementary estimations similar to those used in the proof of Theorem 2.3, we can easily show that:

$$\lim_{n\to+\infty}\left(\prod_{r=1}^{n} f(r)\right)^{1/n} = \lim_{n\to+\infty}\left(\prod_{r=1}^{n} h(r)\right)^{1/n} = 4,$$

which shows in particular that the upper bound of Conjecture 2.5 is optimal.

Now, by exploiting the properties obtained above for the arithmetic function f, we are going to establish some curious properties concerning the 2-adic valuation.

Theorem 2.6 *For all positive integer n and all odd prime number p, we have:*

$$\sum_{r=1}^{n} v_2(r)v_p(r) \ \leq \ \sum_{r=1}^{n} v_p(r) - \left\lfloor \frac{\log n}{\log p} \right\rfloor .$$

Proof. Let n be a positive integer and p be an odd prime number. Since (according to Theorem 2.2), the product $\prod_{r=1}^{n} f(r)$ is a multiple of the positive integer $\mathrm{Odd}(\mathrm{lcm}(1, 2, \ldots, n))$ whose the p-adic valuation is equal to $\lfloor \frac{\log n}{\log p} \rfloor$, then we have:

$$v_p\left(\prod_{r=1}^{n} f(r)\right) = \sum_{r=1}^{n} v_p\left(f(r)\right) \ \geq \ \left\lfloor \frac{\log n}{\log p} \right\rfloor .$$

But by the definition of f, we have for all $r \geq 1$:

$$v_p(f(r)) \ = \ (1 - v_2(r))v_p(r).$$

So, it follows that:

$$\sum_{r=1}^{n} (1 - v_2(r))v_p(r) \geq \left\lfloor \frac{\log n}{\log p} \right\rfloor ,$$

which gives the inequality of the theorem. ■

Theorem 2.7 *Let n be a positive integer and let $a_0 + a_1 2^1 + a_2 2^2 + \cdots + a_s 2^s$ be the representation of n in the binary system. Then we have:*

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = \sum_{i=1}^{s} i a_i.$$

In particular, we have for all $m \in \mathbb{N}$:

$$\sum_{r=1}^{2^m} \frac{v_2(r)(3 - v_2(r))}{2} = m.$$

Proof. By taking the 2-adic valuation in the two hand-sides of the identity (4) and then using (6), we obtain:

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = v_2\left(\prod_{r=1}^{n} h(r)\right) = v_2\left(\frac{n!}{\lfloor\frac{n}{2}\rfloor!\lfloor\frac{n}{4}\rfloor!\lfloor\frac{n}{8}\rfloor!\cdots}\right).$$

It follows by using Legendre's formula (see e.g., [1]) that:

$$\begin{aligned}
\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} &= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^{i+j}} \right\rfloor \\
&= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{u=2}^{\infty} (u-1) \left\lfloor \frac{n}{2^u} \right\rfloor \\
&= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{i=1}^{\infty} i \left\lfloor \frac{n}{2^{i+1}} \right\rfloor .
\end{aligned}$$

By adding to the last series the telescopic series $\sum_{i=1}^{\infty}\left((i-1)\left\lfloor\frac{n}{2^i}\right\rfloor - i\left\lfloor\frac{n}{2^{i+1}}\right\rfloor\right)$ which is convergent with sum zero, we derive that:

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = \sum_{i=1}^{\infty} i \left(\left\lfloor \frac{n}{2^i} \right\rfloor - 2 \left\lfloor \frac{n}{2^{i+1}} \right\rfloor \right).$$

But according to the representation of n in the binary system, we have:

$$\left\lfloor \frac{n}{2^i} \right\rfloor - 2 \left\lfloor \frac{n}{2^{i+1}} \right\rfloor = \begin{cases} a_i & \text{for } i = 1, 2, \ldots, s \\ 0 & \text{for } i > s \end{cases}.$$

Hence

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = \sum_{i=1}^{s} i a_i,$$

as required.
The second part of the theorem is nothing else an immediate application of its first part with $n = 2^m$. The proof is finished. ∎

References

[1] G.H. Hardy and E.M. Wright. The Theory of Numbers, fifth ed., Oxford Univ. Press, London, 1979.